SENIOR RESEARCH

Topic: An Optimal Control Model of Eco-Taxation in the Energy Sector

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Abstract

This paper presents a simple optimal control model for an energy sector that converts fossil fuels into electricity. The goal is to illustrate that the *optimal eco-tax level* on fossil fuel usage is equal to the shadow price of the pollution externality. We develop the model, solve the optimization problem, and interpret the results, showing how the Pigouvian eco-tax internalizes the negative externality by aligning private costs with social costs.

1 Introduction

In the 21st Century, Climate change and environmental sustainability have taken the front stage in global issues. A push towards taking action has prompted nations worldwide to develop policies such as the Paris Agreement 2015, moving the world economy towards zero carbon emissions. The agreement aims to establish a goal for reducing greenhouse gas emissions to maintain the global temperature substantially increases to below 2 degrees Celsius relative to pre-industrial levels and pursue efforts to limit to 1.5 Celcius above the pre-industrial level, recognizing that this would signiEicantly reduce the risks and impacts of climate change (United Nations, 2024). To reduce emissions, economists agree that implementing a price on emissions through mechanisms like cap and trade and carbon taxation is the most effective method for diminishing carbon emissions via market-based incentives (Gucler et al., 2023). A carbon tax aims to put a price on carbon emissions, providing a clear price signal to producers and consumers about their carbon footprint. The additional cost will reduce demand and increase the value proposition for investment in clean generation, transmission, and grid-scale energy storage, prompting producers to innovate and move towards a production process without producing greenhouse gases (Olsen et al., 2018). Consumers will, in turn, be more incentivized to buy environmentally sustainable goods. Shadow pricing in the same vein is also a way to predict the cost of carbon. emissions, and understanding these concepts can lead to more effective policy decisions in reducing GHG emissions.

A great example of an effective eco-tax is Sweden, with a 30 Year of carbon tax history. Sweden possesses one of the highest carbon taxes in the world, initially set at $26 per ton of CO2, escalating to $126 per ton of CO2 in 2020. Despite the high rate, Sweden demonstrated that it can achieve signiEicant emission reduction and strong economic growth. Between 1990 and 2019, Sweden's real GDP per capita increased by almost 50 percent while decreasing greenhouse gas emissions by 27 percent, demonstrating that environmental taxation does not inherently hinder economic progress. (Jonsson et al, 2020).

The energy sector is deEined by the S&P500's global industry ClariEication Standard's indexes deEine the energy sector as consisting of 2 industries: "energy equipment and services" and "Oil, gas, and consumable fuel". According to the United States Environmental Protection Agency (n.d.), energy generation accounted for 34 percent of Global greenhouse gas emissions in 2019, consisting of burning coal, natural gas, and oil for electricity and heating. Industrial usage also accounted for 24 percent of Global greenhouse gas emissions, primarily from fossil fuel burned on site at energy facilities, as well as emissions from chemical, metallurgical, and mineral transformation processes not associated with energy consumption and waste management activities. The energy sector, consisting of these two industries, is responsible for a combined emission of 68 percent of global emissions. This means that if we, as an international community, want to have the highest impact on Greenhouse gas emission reduction, we must start using policies such as a carbon tax to target and transform the energy sector.

2 Literature review

The concept of Shadow Price in environmental economics has been mainly explored regarding the shadow price of 2 variables. Shadow price of energy, and in terms of emissions abatement (Pollutant). The shadow price of a pollutant's standard deEinition is forgone revenue from reducing emissions by one unit (Rodseth, 2023), while the shadow price of energy is deEined as the willingness to pay for an additional unit of energy input to contribute to production (Khadsemvatani and Gordon, 2013). Therefore, shadow prices are not the market price of energy. A carbon tax is a direct monetary charge imposed on fossil fuels based on their carbon content or emissions (Zhunussova, 2022). It is one of the most effective policies to combat climate change. Pricing the emission of greenhouse gases is a well-established approach to internalizing negative externality, shifting the demand-supply equilibrium to a social optimum point (Olsen et al., 2018). Since shadow prices can quantify the impact of solutions or policies, they are helpful for communication from modelers to decision makers (Schwaeppe et al., 2024). This makes shadow prices and carbon tax a great connection and comparison from theory to the real world. The relationship has sparked research and papers to expand and assess these variables.

To understand the relationship between the Shadow price and the Carbon Tax, we must Eirst understand the effects of the carbon tax. Carbon Tax is a policy that can affect the market price of energy because taxes add to the cost of production, and fuel suppliers will typically pass the tax on to consumers in terms of higher prices (IMF F&D, 2019). Empirical Eindings from The Clean Energy Bill of 2012-2014, utilizing a natural experiment based on Australia's carbon pricing mechanism (CPM), demonstrate that wholesale electricity costs rise by 22.1% to 68.0% throughout interconnected regions after accounting for relevant factors. (Wong & Zhang, 2021). Tax policies can be used as an environmental tool to alter energy prices and adjust shadow prices relative to the price of energy (Khadsemvatani and Gordon, 2013). When the shadow price of energy does not match its market price, private enterprise will reallocate energy and other inputs in the production process to maximize proEit (Sheng et al., 2015). This can create energy inefEiciencies and market failure, and the Carbon tax can be used to balance the market price and the shadow price of energy.

Many papers have found similar results regarding carbon taxes. A paper assessing the effects of the unilateral carbon tax on the British power sector from 2013 to 2015 found that the carbon tax led to a signiEicant 38.6 MTCO2 reduction, accounting for 60 percent of the 43 percent emission reduction within those 3 years. Notably, coal-based emissions were reduced by 40 percent during that period, and a shift to renewables as a more carbon-efEicient generation, such as gas emissions, saw a slight rise (Gucler et al., 2023). This means the carbon tax also has the additional effect of encouraging alternative energy sources such as renewables by making them more cost-competitive (Shahzad, 2020). Nevertheless, the carbon tax's effectiveness depends on setting an optimum price equal to the shadow price, which reElects true social cost. Setting too low fails to incentivize behavioral change (Samuel, 2025). Therefore, A low level of carbon tax rate cannot fully reveal the true social cost (Ding, 2022). However, the current carbon tax policies are currently insufEicient. Jain and Kumar (2018) reElected that India's clean energy cess of US$6.5 per ton of coal and US$3.81 per ton of CO2 is inadequate to drive emission reduction to meet their emission reduction pledge for the Paris Agreement.

A carbon price can also restore efEicient resource allocation without needing heavy-handed government interference in individual decisions made by Eirms (Mankiw, 2009). Sheng et al. (2015) used a non-parametric shadow pricing approach to examine energy inefEiciencies in China's 30 provinces from 1998 to 2011. The paper shows that the shadow prices of more than half the provinces are signiEicantly higher than the market price due to inefEiciency in energy allocation. Findings calculated that energy input could be decreased by 36% without reducing output. Furthermore, the paper suggests that energy efEiciency in China stayed stagnant until the signiEicant increase in stricter environmental policies implemented in 2006, which improved energy efEiciency, implying that environmental regulation can help improve energy efEiciency (Wang & Liang, 2022). The author also said that a change in Tax policy can raise the shadow price of energy to match market prices. In India, the 56 coal-Eired thermal power plants are estimated to be able to reduce emission intensity by 16 to 23 percent if they operate efEiciently (Jain & Kumar, 2018). The carbon tax more signiEicantly impacts old and relatively inefEicient Coal plants because they experience a more substantial increase in relative marginal cost, reducing their output (Gucler et al., 2023), so a carbon tax can increase the overall energy generation efEiciency.

Various other papers have also investigated shadow pricing in other countries using different models. Streimikis et al. (2024) assess the cost of sustainable energy use in agriculture using shadow pricing and data envelopment analysis as part of the European Green Deal. Different shadow pricing methods and assumptions can drastically vary the results of shadow price estimation, raising concerns about shadow pricing as a policy tool. While the Carbon Tax offers a more stable and predictable emission control mechanism, it lacks the sector-speciEic adaptability that shadow pricing provides (Schwerhoff et al., 2022). The Eindings show that a one-size-Eits-all carbon tax can be ineffective due to speciEic variations in shadow price and abatement cost.

In addition, many external factors must also be considered before implementing a carbon tax. An empirical study on Taiwan's gasoline market found that under simulation, if a carbon tax of US$20 per ton were introduced, prices would increase by approximately 2.11%, representing a reduction in CO2 emission of 14 million tons (Chiu et al., 2015). The paper illustrated that because Taiwan's market operates under imperfect competition, an oligopoly, carbon tax results in a lower energy price than emission trading, demonstrating the importance of market structure in policy decisions. Evidence from research suggests that a well-designed carbon tax that aligns market price with shadow prices of today and the social value of induced innovation can improve both economic efEiciency and achieve emission reduction (Stiglitz, 2019). Similarly, Nie et al. (2021) analyzed carbon and emission taxes under monopolization. They concluded that the carbon tax is more effective than the emission tax and emphasized the importance of designing the emission tax mechanism under incomplete information. Engstrom and Gars (2015) also highlight how optimal carbon taxation must consider macroeconomic factors such as technological change, spacial differences, and interaction with other Eiscal policies, since energy optimization models are applied to inform decision making and must account for uncertainty as it exists in the real world (Schwaeppe et al., 2024).

Since shadow prices are well deEined in optimization problems (Schwaeppe et al., 2024), papers mainly incorporate shadow prices into creating numerous models to assess the carbon tax optimization. The most prominent is the data envelope analysis. However, there hasn't been a deEinitive best model, and many have undeniable Elaws. Rodseth (2023) found that Data Envelopment Analysis overestimates marginal abatement costs because it overestimates shadow prices faced by inefEicient units, such as in China. The paper found that shadow pricing estimates are signiEicantly higher when accounting for the technical relationship between input and emission. The paper suggests that shadow pricing should focus more on the input-output relationship rather than simply reducing output. Findings indicated that prior economic models have underestimated and call for a reassessment of carbon abatement costs. This means that carbon taxes may not accurately reElect the marginal abatement cost, especially in sectors with complex production technologies (Guo & Prestemon, 2025). Therefore, A novel shadow pricing approach focusing on the dual of the restricted proEit function can, for example, be created from a dual formulation of the proEit function, which may yield more accurate results of shadow pricing (Rodseth 2023).

In this paper, we aim to build a simple optimal control model and Eind the optimized eco-tax level using the derived proEit function. In section 3, we outline key variables and parameters as well as the objective function. In section 4, we set up a Hamiltonian equation and, using Pontryagin's maximum principle, we solve for the marginal beneEit of fossil fuel use (ProEit Function). In section 5, using the derived proEit function, we can interpret the shadow prices and quantify how the eco-tax should be set. Afterward, in section 6, we will conduct a sensitivity test using the model to test the key parameters' effects on pollution produced, fossil fuel use, and eco-tax. The Einding will give insightful theoretical evidence on eco-tax calibration to minimize social damage from pollution.

3 Methodology

In energy economics, a common concern is how to account for the negative externalities of fossil fuel usage, such as carbon emissions or other forms of pollution. A standard policy approach is to impose an eco-tax (also known as a Pigouvian tax) that forces producers of fossil fuels (or electricity generated from fossil fuels) to pay for the damages caused by their emissions. One key insight from the theory of externalities is that this eco-tax rate should be set to the *marginal social damage* of emissions.

A Simple optimal control model can be used to Eind the "marginal social damage" corresponding to pollution's *shadow price*. Below, we set up a simple inEinite-horizon model of a social planner who chooses the *rate of fossil fuel usage* to generate electricity, considering both the beneEits of electricity production and the environmental damages from pollution. We then show rigorously that, in equilibrium, the optimal tax is set equal to the *co-state* (shadow price) of pollution times the emission intensity of fossil fuel usage.

3.1 Model Setup (Variables and parameters)

$x(t)$: The **control variable**, representing the rate at which the economy uses fossil fuels at time t. Burning these fuels produces electricity but also causes pollution.

$F(t)$: A **state variable** for the stock of fossil fuels (if we assume a depletable resource) at time t.

$S(t)$: A **state variable** for the stock of pollution (e.g., CO_2 in the atmosphere) at time t.

$\pi(x)$: The **benefit function** (or surplus) from generating electricity using $x(t)$ units of fossil fuel. Assume $\pi'(x) > 0$, $\pi''(x) < 0$.

$\phi(S)$: The **damage function** from the pollution stock $S(t)$. Assume $\phi'(S) > 0$, $\phi''(S) > 0$ to capture that more pollution increases damages at an increasing rate.

$\rho > 0$: The **discount rate** indicates society's preference for present and future beneEits.

$\alpha \geq 0$: The **emission intensity parameter**, indicating how many units of pollution are created per unit of fossil fuel used.

$\sigma \geq 0$: The **natural decay rate** of pollution, accounting for processes that remove or sequester pollution over time.

$F_0 > 0$: The **initial stock** of fossil fuels.

$S_0 \geq 0$: The **initial stock** of pollution.

3.2 Dynamics

We specify the following evolution (differential equations) for the two state variables:

$$\dot{F}(t) = -x(t), \qquad F(0) = F_0, \tag{1}$$

$$\dot{S}(t) = \alpha x(t) - \sigma S(t), \qquad S(0) = S_0. \tag{2}$$

Interpretation:

Each unit of $x(t)$ directly depletes the fossil fuel stock $F(t)$ at the same rate, hence $\dot{F}(t) = -x(t)$.

Each unit of $x(t)$ emits intensity α units of pollution, but the pollution stock $S(t)$ decays (or is absorbed) at rate σ, hence $\dot{S}(t) = \alpha x(t) - \sigma S(t)$.

3.3 Objective function

A social planner wishes to maximize the net present value of beneEits from electricity generation, minus the disutility (damages) from the resulting pollution:

$$\max_{x(\cdot)\geq 0} \int_0^\infty e^{-\rho t} \Big[\pi\big(x(t)\big) - \phi\big(S(t)\big)\Big]\, dt. \tag{3}$$

4 Solving the model: Pontryagin's maximum principle

4.1 The current-value Hamiltonian

To solve this inEinite-horizon problem, we use the Pontryagin Maximum Principle (PMP). We deEine two costate variables:

$\lambda_F(t)$ for the fossil fuel stock $F(t)$,

$\lambda_S(t)$ for the pollution stock $S(t)$.

These shadow prices represent the marginal value (to the objective function) of having one more unit of F or S, respectively, at time t. The objective function is a variable you want to maximise or minimize, such as proEit or utility. This means $\lambda_F(t)$ can represent when saving one unit of fossil fuel stock (Increasing one unit of fossil fuel stock), how much proEit changes, assuming you want to maximise proEit.

The **current-value Hamiltonian** is:

$$\mathcal{H}\big(F, S, x, \lambda_F, \lambda_S\big) = \underbrace{\big[\pi\big(x(t)\big) - \phi\big(S(t)\big)\big]}_{\text{instantaneous payoff}} + \underbrace{\lambda_F(t)\big[-x(t)\big]}_{\text{effect on } F} + \underbrace{\lambda_S(t)\big[\alpha\, x(t) - \sigma\, S(t)\big]}_{\text{effect on } S}. \quad (4)$$

4.2 First-order conditions (FOCs)

1. **Optimality condition for** $x(t)$**.** Differentiate H with respect to x:

$$\frac{\partial \mathcal{H}}{\partial x} = \pi'(x(t)) \; - \; \lambda_F(t) \; + \; \alpha\, \lambda_S(t) = 0.$$

Thus, the condition for an interior optimum is:

$$\pi'(x(t)) = \lambda_F(t) - \alpha\lambda_S(t). \quad (5)$$

This states that the *marginal beneCit* of fossil fuel usage, $\pi'(x)$, must equal the net shadow cost of fossil fuel use. That net cost has two components:

$\lambda_F(t)$: the *scarcity value* (opportunity cost) of using up one more unit of $F(t)$.

$-\alpha\lambda_S(t)$: the *pollution cost*, which enters with a negative sign here because $\alpha\lambda_S(t)$ is *marginal* external damage for each additional $x(t)$, and thus reduces the net social payoff.

2. **Costate equations.** The costate equations in *current value* form are:

$$\dot{\lambda}_F(t) = \rho\, \lambda_F(t) - \frac{\partial \mathcal{H}}{\partial F} = \rho\, \lambda_F(t), \quad \text{(since } \partial \mathrm{H}/\partial F = 0\text{)}, \quad (6)$$

$$\dot{\lambda}_S(t) = \rho\, \lambda_S(t) - \frac{\partial \mathcal{H}}{\partial S} = \rho\, \lambda_S(t) \; - \; \Big[-\phi'(S(t)) \; + \; \lambda_S(t)\,(-\sigma)\Big]. \quad (7)$$

To see (7) more clearly:

$$\frac{\partial \mathcal{H}}{\partial S} = \underbrace{-\phi'(S)}_{\text{derivative w.r.t. } S \text{ in payoff}}, \quad + \underbrace{\lambda_S \left(\frac{\partial}{\partial S}[\alpha x - \sigma S]\right)}_{=\lambda_S(-\sigma)}$$

so

$$\dot{\lambda}_S(t) = \rho\lambda_S(t) + \phi'(S(t)) - \sigma\lambda_S(t) = (\rho + \sigma)\lambda_S(t) + \phi'(S(t)).$$

The costate equations describe how the shadow price of state variables changes over time in an optimum extraction problem. $\rho\lambda_F(t)$ represents the Discounting effect since society values future fossil fuel availability less than in the present. At the same time, the Hamiltonian differential captures how state variable changes can affect the objective function.

3. **Transversality conditions.** For an inEinite-horizon problem with well-behaved solutions, we typically have

$$\lim_{t\to\infty} e^{-\rho t}\lambda_F(t)F(t) = 0, \quad \lim_{t\to\infty} e^{-\rho t}\lambda_S(t)S(t) = 0.$$

These ensure no inEinite-value terminal stocks.

5 Interpretation of the shadow prices and the eco-tax

In the social planner's solution, the Eirst-order condition (5),

$$\pi'(x(t)) = \lambda_F(t) - \alpha\lambda_S(t),$$

balances marginal beneEit and marginal social cost. Now, suppose that we want to see how this solution can be decentralized *via* a market mechanism.

Decomposing $\lambda_F(t)$ and $\lambda_S(t)$: private vs. external costs

Scarcity rent (private cost): $\lambda_F(t)$ is the shadow price of depleting one more unit of the fossil fuel stock. If the fossil fuel resource is privately owned (or if there is a well-functioning market for the resource), a proEit-maximizing extractor would already incorporate $\lambda_F(t)$ into its decisions. In effect, $\lambda_F(t)$ is an internal cost (the so-called *Hotelling rent*) that the owner of the resource will charge or at least recognize in an efEicient resource market.

Pollution shadow price (external cost): $\lambda_S(t)$ is the shadow price associated with the *pollution stock* $S(t)$. This is an external cost: without environmental regulation, individual Eirms do *not* pay for the damage caused by increasing $S(t)$.

Thus, in a well-functioning market for the resource itself, private agents would face a marginal cost of $\lambda_F(t)$ (reElecting scarcity). However, they would *not* face the cost $\alpha\lambda_S(t)$, which represents the social cost of the pollution externality.

Why the Pigouvian tax is $\alpha\lambda_S(t)$

To ensure that private decision-makers (e.g. electricity producers) *also* take pollution damages into account, a regulator can impose an *eco-tax* $\tau(t)$ per unit of $x(t)$. Under such a tax, a competitive Eirm sees the following *private* marginal condition:

$$\pi'(x(t)) = \underbrace{(\text{price of fossil fuel resource})}_{\lambda_F(t)\text{ if resource is privately owned}} + \underbrace{(\text{tax})}_{\tau(t)}.$$

The equation represents the Economy's marginal proEit as equal to the beneEit from burning a unit of fossil fuel, quantiEied monetarily by the price of fossil fuel resource and the beneEit from the tax imposed by a regulator to internalize social damage in the private sector.

If the resource market is perfectly competitive and fully internalizes $\lambda_F(t)$ as the scarcity cost, then the only missing piece is the pollution externality. Therefore, for the *socially optimal* choice of $x(t)$, we must have

$$\pi'(x(t)) = \lambda_F(t) + \tau(t).$$

Comparing this to the social planner's FOC,

$$\pi'(x(t)) = \lambda_F(t) - \alpha\lambda_S(t),$$

we require

$$\lambda_F(t) + \tau(t) = \lambda_F(t) - \alpha\lambda_S(t).$$

Hence

$$\tau(t) = -\alpha\lambda_S(t). \tag{8}$$

The negative sign in front of $\alpha\lambda_S(t)$ in the planner's FOC (equation (5)) reElects that *pollution is a cost and that a tax must be collected from producers and will decrease proCitability of private producers*. When we bring that term to the left-hand side for the private agent, it *adds* to the private agent's cost in the form of a positive tax. Thus, the per-unit tax $\tau(t)$ that internalizes the externality is precisely $\alpha\lambda_S(t)$ (with the negative sign accounted for in the rearrangement).[1]

This *proves* that the Pigouvian tax *must* be set to the marginal external damage:

$$\text{Marginal External Damage} = \alpha\lambda_S(t).$$

Since $\lambda_S(t)$ is the social shadow price of pollution stock, $\alpha\lambda_S(t)$ is the additional social cost per unit of $x(t)$, i.e. the social damage from the extra pollution caused by burning one unit of fossil fuel.

Interpretation

Scarcity rent $\lambda_F(t)$ is internal to private agents *if* property rights over the resource are well-deEined. In other words, if a Eirm must pay $\lambda_F(t)$ per unit of fossil fuel to the resource owner, it already takes scarcity into account.

Pollution cost $\alpha\lambda_S(t)$ is *not* internal to private agents unless an explicit mechanism forces it to be. A Pigouvian tax of exactly $\tau(t) = \alpha\lambda_S(t)$ per unit of fuel ensures that the private user bears the social cost of its pollution, thereby "internalizing" the externality.

Conclusion: The optimal tax rate $\tau(t)$ is exactly the product of the emission intensity α and the pollution shadow price $\lambda_S(t)$. If $\alpha = 1$, then $\tau(t) = \lambda_S(t)$.

6 Simulations

In this section, we numerically simulate the dynamics of the system to illustrate:

1. How the state variables $F(t)$ and $S(t)$ evolve over time when the optimal control $x(t)$ is used.

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[1] If one prefers a more direct reading: in the planner's FOC, the net social cost is $\lambda_F - \alpha\lambda_S$. Rearranging to see what extra cost a private agent must face to replicate that FOC, the wedge is indeed $\alpha\lambda_S$.

2. How the shadow price $\lambda_S(t)$ (and thus the optimal eco-tax $\tau(t) = \alpha\lambda_S(t)$) evolves.

3. How sensitive the outcomes (especially the eco-tax path) are to changes in key parameters α, σ, and ρ.

6.1 Simulation parameters

To run a concrete simulation, we must choose speciEic functional forms and parameter values. For simplicity, let us pick:

Linear-quadratic beneEits/damages for easier interpretation:

$$\pi(x) = a\,x - \frac{b}{2}\,x^2 \quad \Longrightarrow \quad \pi'(x) = a - b\,x.$$

Quadratic pollution damage:[2]

$$\phi(S) = \frac{d}{2}\,S^2 \quad \Longrightarrow \quad \phi'(S) = d\,S.$$

State equations:

$$\dot{F}(t) = -x(t), \dot{S}(t) = \alpha x(t) - \sigma\, S(t).$$

Costate equations (current value):

$$\dot{\lambda}_F(t) = \rho\lambda_F(t),$$

$$\dot{\lambda}_S(t) = (\rho + \sigma)\lambda_S(t) + dS(t).$$

From the FOC $\pi'(x) = a - bx = \lambda_F - \alpha\lambda_S$, we solve for

$$x(t) = \frac{a - \lambda_F(t) - \alpha\,\lambda_S(t)}{b}.$$

Let us specify some numerical values for a baseline scenario:

$$a = 10, \qquad b = 2,$$

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[2] One could easily change to a different damage function, e.g. $\phi(S) = dS^\gamma$, as long as it is increasing and convex in S.

$$d = 0.05, \qquad \alpha = 0.2,$$
$$\rho = 0.03, \qquad \sigma = 0.02,$$
$$F_0 = 100, \qquad S_0 = 10.$$

We choose a moderate discount rate ρ = 3%, a small decay rate σ = 2%, and an emission intensity $\alpha = 0.2$. The resource is initially quite large ($F_0 = 100$), and the initial pollution stock is $S_0 = 10$.

6.2 Simulation results for different emission intensity levels

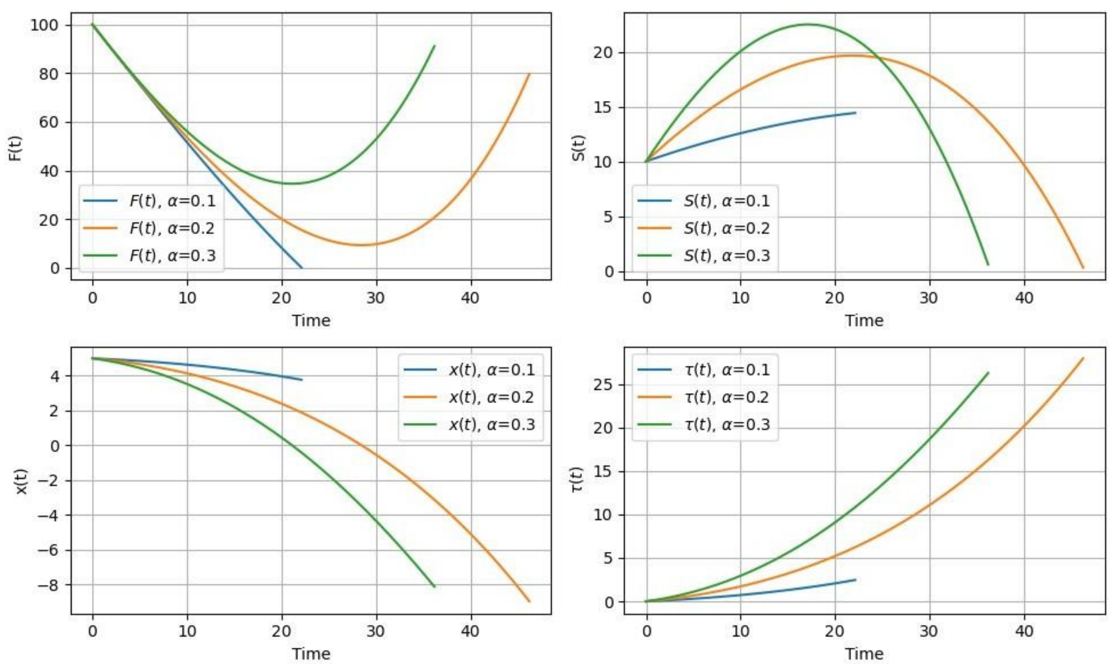

Figure 1: Illustrative simulation results for the baseline parameters.

Figure 1 shows how the resource stock $F(t)$, the pollution stock $S(t)$, the usage rate $x(t)$, and the eco-tax $\tau(t)$ evolve over time for different levels of emission intensity $\alpha = 0.1, 0.2$, and 0.3. The main insights from these results are:

Resource stock $F(t)$ (*U-shaped*): Initially, the resource stock declines as the planner extracts fossil fuel. However, as the usage rate slows down over time, the stock stabilizes and even starts increasing again, forming a U-shaped curve. When α is higher (green curve), the resource is extracted more aggressively at Eirst, leading to a steeper decline, but later recovers more strongly. In contrast, lower α (blue curve) results in a more gradual decline and a gentler recovery.

Pollution stock $S(t)$ (*inverted U-shaped*): Pollution initially increases as fossil fuels are burned. However, as extraction slows down and natural decay removes pollution, the stock of pollution starts decreasing, creating an inverted U-shape. Higher α causes pollution to rise more sharply at Eirst because each unit of fuel burned produces more emissions. However, the stronger reduction in usage later on means that pollution also decreases more rapidly toward the end of the time horizon.

Usage rate $x(t)$ (*gradual decline*): The planner starts with high fuel usage but reduces it over time to manage both resource depletion and pollution costs. This creates a smooth, downward-sloping curve. When α is larger, the planner initially extracts more but cuts back more aggressively later. With lower α, the reduction in fuel use happens more gradually.

Eco-tax $\tau(t)$ (*increasing over time*): The tax on emissions starts low but rises as pollution costs become more signiEicant. Since the eco-tax is proportional to pollution damages, it increases more rapidly when α is higher. The tax follows an upward-curving path, meaning that the cost of emitting rises more sharply as time progresses.

Effect of different α **values.** Comparing the three cases ($\alpha = 0.1, 0.2, 0.3$), we observe that higher α:

Leads to a **steeper initial drop** in $F(t)$ followed by a stronger rebound.

Produces a **higher peak** in $S(t)$ before pollution starts to decline.

Causes **higher initial fuel usage** but also a more rapid reduction in extraction later.

Results in a **higher and faster-growing** eco-tax as pollution damages increase.

Overall, increasing the emission intensity α makes short-term resource extraction more aggressive, but this forces a stronger cutback in usage later. Pollution rises more sharply at Eirst but then falls more quickly, and the eco-tax becomes steeper to discourage further emissions.

6.3 Sensitivity analyses

We now explore how changing two key parameters—the pollution decay rate σ, and the discount rate ρ—affects the behavior of our model. Figures 2 and 3 show the trajectories of the resource stock $F(t)$, the pollution stock $S(t)$, the usage rate $x(t)$, and the eco-tax $\tau(t)$ when we vary each parameter around its baseline value.

Varying the pollution decay rate σ

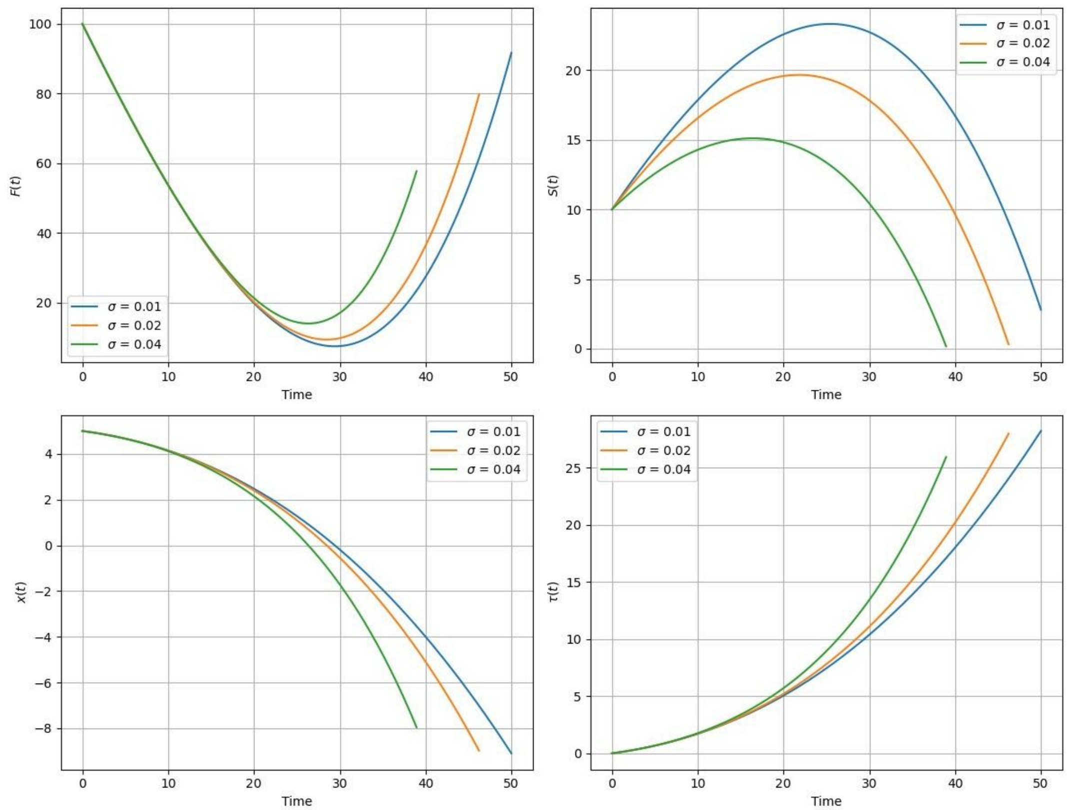

Figure 2: Sensitivity analysis: changes in σ.

Figure 2 shows how the system behaves when σ = 0.01,0.02, and 0.04. The main observations are:

Resource stock $F(t)$: A higher σ means pollution dissipates faster, so the planner can allow slightly higher usage without letting pollution get too large. As a result, the resource stock tends to dip more at Eirst but also rebounds strongly, ending at a higher level.

Pollution stock $S(t)$: With faster decay (larger σ), the pollution peak is lower and occurs earlier. Over time, $S(t)$ falls more quickly, so we see a gentler inverted U-shape that ends at a smaller Einal level.

Usage rate $x(t)$: Because pollution is less of a threat when it decays faster, the planner can afford to use more fossil fuel early on. Hence, the curve for a higher σ starts out above the others, but it still declines over time as the resource eventually becomes costly or the remaining pollution cost accumulates.

Eco-tax $\tau(t)$: When σ is larger, pollution never builds up as much, so the shadow price of pollution is lower. Consequently, the tax grows more slowly and stays below the paths for lower σ.

In summary, a larger pollution decay rate σ reduces the overall pollution burden, allowing more fuel usage early on and lowering the eco-tax path.

Varying the discount rate ρ

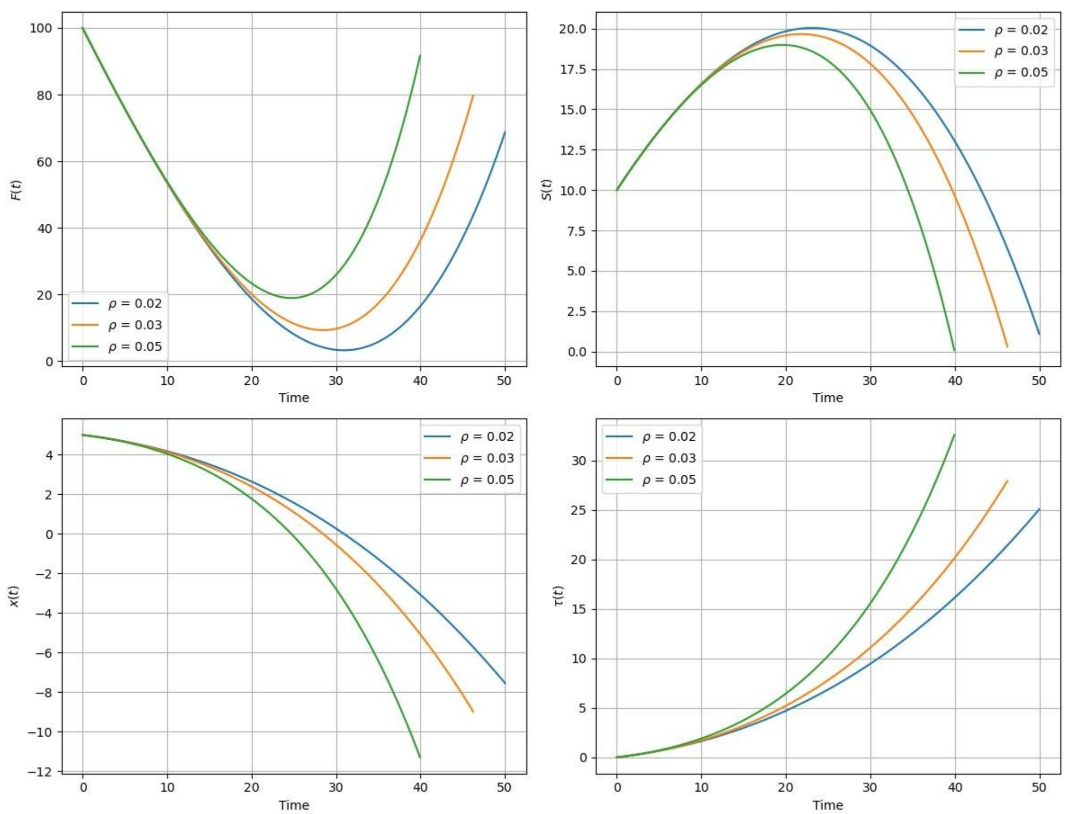

Figure 3: Sensitivity analysis: changes in ρ.

Figure 3 illustrates what happens when we change ρ. We see:

Resource stock $F(t)$: With a higher discount rate, the planner places more weight on immediate beneEits, so extraction is front-loaded. This leads to a deeper initial decline in $F(t)$ but also a more pronounced rebound later, again giving the curve a U-shape.

Pollution stock $S(t)$: The quicker extraction early on raises pollution sooner, resulting in a higher peak. However, once the resource use drops off, pollution decays to a lower Einal level for higher ρ.

Usage rate $x(t)$: A larger ρ pushes the planner to exploit the resource earlier, so the usage curve starts out higher. Over time, $x(t)$ falls more sharply once immediate gains are exhausted and pollution costs grow.

Eco-tax $\tau(t)$: With heavier discounting of future damages, the tax remains relatively low at Eirst. Even though it rises over time, it does so from a smaller base compared to lower discount rates. Ultimately, the higher ρ path may still end up lower, reElecting the planner's weaker concern for future pollution.

Thus, increasing ρ speeds up early extraction, drives an earlier and higher pollution peak, and delays a signiEicant rise in the eco-tax.

Overall insights

Across all three sensitivity analyses, the shapes of $F(t)$ (U-shaped), $S(t)$ (inverted Ushaped), $x(t)$ (downward-sloping), and $\tau(t)$ (upward-sloping) remain broadly consistent. What changes is *how steeply* each curve rises or falls, and *where* it ends up by the Einal time. In particular:

α (emission intensity) mainly affects how much pollution is created per unit of fossil fuel, making the planner either more or less aggressive in managing usage and tax.

σ (pollution decay) inEluences how quickly pollution dissipates, letting the planner adjust extraction and taxes based on how fast the environment recovers.

ρ (discount rate) determines how much the planner values present vs. future costs, leading to different timing for extraction, pollution peaks, and tax growth.

Overall, these results conEirm that higher α raises pollution costs, higher σ reduces them, and higher ρ favors earlier resource usage. The eco-tax and usage paths adjust in each scenario to balance the trade-offs between current beneEits and future damages.

These simulation results conEirm our theoretical predictions:

Eco-tax equals $\alpha\lambda_S(t)$: In every simulation, the per-unit eco-tax is exactly the product of the emission intensity α and the shadow price of pollution $\lambda_S(t)$. This conEirms that the tax internalizes the external cost of pollution.

Resource usage and pollution dynamics: Over time, the usage rate $x(t)$ declines. This happens both because the cost of depleting the resource (measured by the scarcity rent $\lambda_F(t)$) increases and because the external pollution cost (reElected in $\lambda_S(t)$) rises when pollution accumulates. As a result, the pollution stock $S(t)$ initially rises, reaches a peak, and then falls.

Higher emission intensity α: When α increases, each unit of fossil fuel causes more pollution. The simulations show that higher α leads to a higher eco-tax path, a sharper initial extraction (or more front-loaded usage), and then a more aggressive cutback in $x(t)$. This results in a deeper dip and stronger rebound in $F(t)$ and a steeper, more peaked $S(t)$ proEile.

Faster pollution decay σ: With a larger σ, pollution dissipates faster. The simulations indicate that a faster decay rate keeps the pollution stock lower and slows the rise of $\lambda_S(t)$, which in turn keeps the eco-tax lower over time.

Higher discount rate ρ: A higher ρ means that the planner puts more weight on current beneEits relative to future costs. This leads to more front-loaded resource extraction and a slower initial increase in the shadow prices. As a consequence, while pollution may eventually become signiEicant, the initial eco-tax is lower because future damages are discounted.

Overall, the simulations illustrate how the model's optimal control solution produces speciEic time paths for fossil fuel usage, pollution accumulation, and the eco-tax.

7 Conclusion

In this paper, we developed and analyzed an optimal control model of eco-taxation in the energy sector. The main Eindings of our study can be summarized as follows:

Optimal eco-tax equals marginal external damage: We showed that the optimal eco-tax that a social planner should impose is given by $\tau(t) = \alpha\lambda_S(t)$, where α is the emission intensity and $\lambda_S(t)$ is the shadow price of pollution. This result conEirms the classic Pigouvian taxation principle, which states that the tax should equal the marginal external cost of pollution (Carlton & Loury, 1986).

Dynamics of resource usage and pollution: Our simulation results reveal that the resource stock $F(t)$ exhibits a U-shaped pattern over time, while the pollution stock $S(t)$ follows an inverted U-shape. The usage rate $x(t)$ declines in a smooth, downward-concave manner, and the eco-tax $\tau(t)$ increases along an upward-convex path. These patterns indicate that higher emission intensity leads to a more aggressive early extraction followed by a stronger cutback, which helps to curb long-term pollution.

Sensitivity to key parameters: The model demonstrates how changes in the key parameters affect the system:

- A higher α increases the marginal pollution cost, resulting in a steeper eco-tax and a more front-loaded extraction strategy.
- A higher pollution decay rate σ reduces the buildup of pollution, thereby lowering the eco-tax and allowing for relatively higher early resource usage.
- A higher discount rate ρ places greater emphasis on current beneEits, leading to earlier extraction and a slower initial rise in the eco-tax.

Policy implications: The main policy takeaway is that a well-calibrated eco-tax, set equal to the marginal external damage $\alpha\lambda_S(t)$, can internalize the pollution externality (Ligthart, 1998). By doing so, private Eirms will face the true social cost of fossil fuel use, which encourages more sustainable production and consumption decisions. Policymakers must carefully consider the values of emission intensity, pollution decay, and discount rates when designing such taxes to achieve the desired balance between economic beneEits and environmental protection.

Limitations: While our model provides clear insights into the optimal design of ecotaxes, it is based on a highly simpliEied framework. For instance, the model assumes a single fossil fuel source, deterministic dynamics, and a speciEic functional form for beneEits and damages. These simpliEications mean that the model does not capture all real-world complexities, such as technological changes, multi-sector interactions, or uncertainty in environmental damages.

Directions for future research: Future studies could extend this work in several directions to enhance realism and policy relevance. For example, one could incorporate:

Stochastic elements to account for uncertainty in pollution dynamics or market conditions improving model's robusness. Stochastic integrated assessment models (SIAMs) have showed promise in addressing the unpredictable nature of climatic tipping points and policy responses (Lemoine & Traeger, 2014).

Multiple energy sources and modeling of substitution effects between fossil fuels and renewable energy technologies. Considering these substitute possibilities is essential for comprehending the energy system's responsiveness to policy change and technology advancement (e.g., Gillingham and Sweeney, 2010).

More realistic damage functions or the integration of climate change dynamics including tipping points, feedback loops and regional variation in climate impact. Recent critiques have highlighted the shortcomings of traditional damage functions and the necessity for models that more accurately represent empirical and physical facts. (e.g., Dietz and Stern, 2015; Weitzman, 2009)

Behavioral aspects and heterogeneous agents to capture a wider range of decision-making processes, especially in contexts where bounded rationality, fairness concerns and asymetrical information are pivotal (e.g., Gennaioli and Shleifer, 2018). Modeling heterogeneity facilitates the examination of distributional effects and policy equality, which are widely acknowledged as fundamental to the formulation of climate policy.

Such extensions would help reEine the model's policy recommendations and provide a deeper understanding of how optimal eco-taxation can be implemented in practice.

Overall, this study contributes to the literature by rigorously demonstrating that the optimal eco-tax, designed to internalize externalities in the energy sector, is determined by the marginal external damage. The insights gained here can serve as a helpful benchmark for policymakers aiming to design environmentally effective and economically efEicient tax policies.

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9 Python Code:

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

#######################################
# DeEine a function to solve the ODEs
#######################################
def solve_model(a, b, d, alpha, rho, sigma, F0, S0,
        lambdaF0_guess=0.0,
        lambdaS0_guess=0.0,
        T=50, N=200):
  """
  Solves the ODE system for given parameter values.
  Integration stops when F(t) or S(t) reaches 0.
  Returns t_vals, F_vals, S_vals, x_vals, tau_vals.
  """
  # ODE system
  def odes(t, y):
    F, S, lamF, lamS = y

    # --- FOC: subtract alpha*lamS ---
    x = (a - lamF - alpha * lamS) / b

    dF_dt = -x
    dS_dt = alpha * x - sigma * S
    dLamF_dt = rho * lamF
    dLamS_dt = (rho + sigma) * lamS + d * S
    return [dF_dt, dS_dt, dLamF_dt, dLamS_dt]

  # Event function to stop integration when F(t) reaches zero
  def event_F_zero(t, y):
    return y[0]  # F(t)
  event_F_zero.terminal = True
  event_F_zero.direction = -1  # Trigger only when F is decreasing through 0

  # Event function to stop integration when S(t) reaches zero
```

```
    def event_S_zero(t, y):
        return y[1]  # S(t)
    event_S_zero.terminal = True
    event_S_zero.direction = -1  # Trigger only when S is decreasing through 0

    # Time grid
    t_span = (0, T)
    t_eval = np.linspace(0, T, N)

    # Initial condition vector
    y0 = [F0, S0, lambdaF0_guess, lambdaS0_guess]

    # Solve ODE with event detection
    sol = solve_ivp(
        odes, t_span, y0,
        t_eval=t_eval,
        events=[event_F_zero, event_S_zero],
        dense_output=True
    )

    t_vals = sol.t
    F_vals = sol.y[0]
    S_vals = sol.y[1]
    lamF_vals = sol.y[2]
    lamS_vals = sol.y[3]

    # Compute control and tax variables
    x_vals = (a - lamF_vals - alpha * lamS_vals) / b
    tau_vals = alpha * lamS_vals

    return t_vals, F_vals, S_vals, x_vals, tau_vals

########################################
# Baseline parameter values
########################################
a = 10.0
b = 2.0
d = 0.05
# Baseline parameters for sensitivity analyses
alpha_baseline = 0.2
rho_baseline = 0.03
sigma_baseline = 0.02
F0 = 100.0
S0 = 10.0

########################################
```

```
# Sensitivity Analysis: Emission Intensity (alpha)
#########################################
alpha_list = [0.1, 0.2, 0.3]
results_alpha = {}

for alpha in alpha_list:
    t, Fv, Sv, xv, tauv = solve_model(a, b, d, alpha, rho_baseline, sigma_baseline, F0, S0)
    results_alpha[alpha] = (t, Fv, Sv, xv, tauv)

plt.Eigure(Eigsize=(12,  10))

# Subplot 1: F(t)
plt.subplot(2,2,1)
for alpha in alpha_list:
    t, Fv, _, _, _ = results_alpha[alpha]
    plt.plot(t, Fv, label=f'$α$ = {alpha}')
plt.xlabel('Time')
plt.ylabel('$F(t)$')
plt.legend()
plt.grid(True)

# Subplot 2: S(t)
plt.subplot(2,2,2)
for alpha in alpha_list:
    t, _, Sv, _, _ = results_alpha[alpha]
    plt.plot(t, Sv, label=f'$α$ = {alpha}')
plt.xlabel('Time')
plt.ylabel('$S(t)$')
plt.legend()
plt.grid(True)

# Subplot 3: x(t)
plt.subplot(2,2,3)
for alpha in alpha_list:
    t, _, _, xv, _ = results_alpha[alpha]
    plt.plot(t, xv, label=f'$α$ = {alpha}')
plt.xlabel('Time')
plt.ylabel('$x(t)$')
plt.legend()
plt.grid(True)

# Subplot 4: τ(t)
plt.subplot(2,2,4)
for alpha in alpha_list:
    t, _, _, _, tauv = results_alpha[alpha]
    plt.plot(t, tauv, label=f'$α$ = {alpha}')
```

```
plt.xlabel('Time')
plt.ylabel('$τ(t)$')
plt.legend()
plt.grid(True)

plt.tight_layout(rect=[0, 0.03, 1, 0.95])
plt.show()

######################################
# Sensitivity Analysis: Pollution Decay Rate (sigma)
######################################
sigma_list = [0.01, 0.02, 0.04]
results_sigma = {}

for sigma in sigma_list:
    t, Fv, Sv, xv, tauv = solve_model(a, b, d, alpha_baseline, rho_baseline, sigma, F0, S0)
    results_sigma[sigma] = (t, Fv, Sv, xv, tauv)

plt.Eigure(Eigsize=(12,  10))

# Subplot 1: F(t)
plt.subplot(2,2,1)
for sigma in sigma_list:
    t, Fv, _, _, _ = results_sigma[sigma]
    plt.plot(t, Fv, label=f'$σ$ = {sigma}')
plt.xlabel('Time')
plt.ylabel('$F(t)$')
plt.legend()
plt.grid(True)

# Subplot 2: S(t)
plt.subplot(2,2,2)
for sigma in sigma_list:
    t, _, Sv, _, _ = results_sigma[sigma]
    plt.plot(t, Sv, label=f'$σ$ = {sigma}')
plt.xlabel('Time')
plt.ylabel('$S(t)$')
plt.legend()
plt.grid(True)

# Subplot 3: x(t)
plt.subplot(2,2,3)
for sigma in sigma_list:
    t, _, _, xv, _ = results_sigma[sigma]
    plt.plot(t, xv, label=f'$σ$ = {sigma}')
plt.xlabel('Time')
```

```
plt.ylabel('$x(t)$')
plt.legend()
plt.grid(True)

# Subplot 4: τ(t)
plt.subplot(2,2,4)
for sigma in sigma_list:
    t, _, _, _, tauv = results_sigma[sigma]
    plt.plot(t, tauv, label=f'$σ$ = {sigma}')
plt.xlabel('Time')
plt.ylabel('$τ(t)$')
plt.legend()
plt.grid(True)

plt.tight_layout(rect=[0, 0.03, 1, 0.95])
plt.show()

########################################
# Sensitivity Analysis: Discount Rate (rho)
########################################
rho_list = [0.02, 0.03, 0.05]
results_rho = {}

for rho in rho_list:
    t, Fv, Sv, xv, tauv = solve_model(a, b, d, alpha_baseline, rho, sigma_baseline, F0, S0)
    results_rho[rho] = (t, Fv, Sv, xv, tauv)

plt.Eigure(Eigsize=(12,  10))

# Subplot 1: F(t)
plt.subplot(2,2,1)
for rho in rho_list:
    t, Fv, _, _, _ = results_rho[rho]
    plt.plot(t, Fv, label=f'$ρ$ = {rho}')
plt.xlabel('Time')
plt.ylabel('$F(t)$')
plt.legend()
plt.grid(True)

# Subplot 2: S(t)
plt.subplot(2,2,2)
for rho in rho_list:
    t, _, Sv, _, _ = results_rho[rho]
    plt.plot(t, Sv, label=f'$ρ$ = {rho}')
plt.xlabel('Time')
plt.ylabel('$S(t)$')
```

```
plt.legend()
plt.grid(True)

# Subplot 3: x(t)
plt.subplot(2,2,3)
for rho in rho_list:
    t, _, _, xv, _ = results_rho[rho]
    plt.plot(t, xv, label=f'$ρ$ = {rho}')
plt.xlabel('Time')
plt.ylabel('$x(t)$')
plt.legend()
plt.grid(True)

# Subplot 4: τ(t)
plt.subplot(2,2,4)
for rho in rho_list:
    t, _, _, _, tauv = results_rho[rho]
    plt.plot(t, tauv, label=f'$ρ$ = {rho}')
plt.xlabel('Time')
plt.ylabel('$τ(t)$')
plt.legend()
plt.grid(True)

plt.tight_layout(rect=[0, 0.03, 1, 0.95])
plt.show
```